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Involutive generators and actions for the group Φ_2

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Abstract. The automorphisms of the free group F_2 form the group Φ_2. Three involutions are shown to generate Φ_2. The homomorphism $h_2 : \Phi_2 \to Gl(2, Z)$ induced by the Abelianization $h_1 : F_2 \to Z^2$ is generated by three affine reflections $R^3 \to R^3$. The actions of Φ_2 induced by the homomorphisms $F_2 \to SU(2), SU(1,1), Sl(2,C)$ are generated by algebraic involutions and geometric reflections.

1. The group Φ_2

Nielsen, in 1924 (compare [1,2]), gave generators and relations for the automorphism group Φ_n of the free group F_n. We shall use the symbols Φ_n for the group and $\mathrm{Aut}(F_n)$ for its action on F_n. Let h_1 denote the homomorphism from F_n to the Abelian group Z^n with n generators. This Abelianization yields a second homomorphism $h_2 : \Phi_n = \mathrm{Aut}(F_n) \to Gl(n, Z) = \mathrm{Aut}(Z^n)$. The groups $(Z^n, GL(n, Z))$ are the basis for classical (periodic) crystallography [3]. The groups (F_n, Φ_n) were proposed in [4] as a basis for non-periodic quasicrystallography. Some specific results in this direction were given in [5–7]. We present in sections 1–3 some general results for Φ_2 and in sections 4–6 for its induced action in various real and complex geometries. For applications of these actions in physics we refer to [6, 8–10] and to the references quoted therein.

First we introduce a new set of generators for Φ_2. For $n = 2$, Nielsen showed that Φ_2 has three generators $\langle\sigma, P, U\rangle$ with relations

$$\begin{aligned} &R_1, R_2 : \sigma^2 = P^2 = e \\ &R_3 : (P\sigma PU)^2 = e \\ &R_4 : U^{-1}PUP\sigma U\sigma P\sigma = e \\ &R_5 : U\sigma U\sigma = \sigma U\sigma U. \end{aligned} \tag{1}$$

Note that, with the additional relation $(U\sigma)^2 = e$, Φ_2 reduces to $Gl(2, Z)$ [1].

Define the group G with generators $\langle c_2, c_3, \sigma_1\rangle$ and relations

$$\begin{aligned} &Q_1, Q_2, Q_3 : (c_2)^2 = (c_3)^2 = (\sigma_1)^2 = e \\ &Q_4 : (c_2c_3)^3 = e \\ &Q_5 : (c_3\sigma_1)^4 = e \\ &Q_6 : \sigma_1c_2(\sigma_1c_3)^2c_2\sigma_1 = c_2(\sigma_1c_3)^2c_2. \end{aligned} \tag{2}$$

Note from $Q_1 - Q_5$ that $\langle c_2, c_3\rangle$ and $\langle c_3, \sigma_1\rangle$ generate Coxeter groups.

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Proposition 1. The groups Φ_2 and G are isomorphic.

Proof. An easy calculation shows that the isomorphism is established by setting

$$\sigma = \sigma_1 \qquad P = \sigma_1 c_3 \sigma_1 \qquad U = c_3 \sigma_1 c_3 c_2 \tag{3}$$

with the inverse

$$\sigma_1 = \sigma \qquad c_3 = \sigma P \sigma \qquad c_2 = \sigma P \sigma P \sigma U. \tag{4}$$

The sets of relations $R_1, \ldots, R_5$ and $Q_1, \ldots, Q_6$ become equivalent. □

Proposition 2. The subgroup $\mathcal{F}_2$ generated by

$$t_1 := (c_3 c_2 \sigma_1 c_3 \sigma_1)^2$$
$$t_2 := (\sigma_1 c_3 \sigma_1 c_3 c_2)^2 \tag{5}$$

is normal in Φ_2.

Proof. It suffices to conjugate $\langle t_1, t_2 \rangle$ with the generators of Φ_2. We obtain

$$c_2 t_1 c_2 = t_1 t_2 \qquad c_2 t_2 c_2 = t_2^{-1}$$
$$c_3 t_1 c_3 = t_2^{-1} \qquad c_3 t_2 c_3 = t_1^{-1} \tag{6}$$
$$\sigma_1 t_1 \sigma_1 = t_1^{-1} \qquad \sigma_1 t_2 \sigma_1 = t_2.$$

The last two results require the use of relation Q_6. □

There is a second normal subgroup of Φ_2:

Proposition 3. The subgroup $\mathcal{H}_2$ generated by the involutions

$$q_1 = c_2 (\sigma_1 c_3)^2 c_2 \qquad q_2 = c_3 c_2 (\sigma_1 c_3)^2 c_2 c_3 \qquad q_3 = (\sigma_1 c_3)^2 \tag{7}$$

is normal in Φ_2. The normal subgroup $\mathcal{F}_2$ is the subgroup of $\mathcal{H}_2$ generated by an even number of involutions.

Proof. For the normal property we conjugate $\langle q_1, q_2, q_3 \rangle$ with the generators of Φ_2 to obtain, with the help of $Q_1 \ldots Q_6$,

$$c_2 q_1 c_2 = q_3 \qquad c_2 q_2 c_2 = q_2 \qquad c_2 q_3 c_2 = q_1$$
$$c_3 q_1 c_3 = q_2 \qquad c_3 q_2 c_3 = q_1 \qquad c_3 q_3 c_3 = q_3 \tag{8}$$
$$\sigma_1 q_1 \sigma_1 = q_1 \qquad \sigma_1 q_2 \sigma_1 = q_3 q_2 q_3 \qquad \sigma_1 q_3 \sigma_1 = q_3.$$

This shows the normal property of $\mathcal{H}_2$. The normal subgroup and even property of $\mathcal{F}_2$ follow from the relations

$$t_1 = q_2 q_3 \qquad t_2 = q_3 q_1. \tag{9}$$

□

2. Φ_2 and the geometry of automorphisms of F_2

To recognize Φ_2 as the group Aut(F_2) we must give its action on F_2. By using the prescription due to Nielsen [1] and equation (4) we obtain for the images $\langle y_1, y_2\rangle$ of $\langle x_1, x_2\rangle$, under the generators of Φ_2, the following transformations

$$\begin{aligned} c_2 &: \quad y_1 = x_1 x_2 \qquad y_2 = x_2^{-1} \\ c_3 &: \quad y_1 = x_2^{-1} \qquad y_2 = x_1^{-1} \\ \sigma_1 &: \quad y_1 = x_1^{-1} \qquad y_2 = x_2. \end{aligned} \tag{10}$$

The group $A_2 \sim \langle c_2, c_3\rangle < \Phi_2$ generalizes for F_n into the non-commutative Coxeter group $A_n\langle\Phi_n$. Consider now the images y_1, y_2 for the elements in equation (5) of Φ_2:

$$\begin{aligned} t_1 = (c_3c_2\sigma_1c_3\sigma_1)^2 &: \quad y_1 = x_1 \qquad y_2 = x_1^{-1}x_2x_1 \\ t_2 = (\sigma_1c_3\sigma_1c_3c_2)^2 &: \quad y_1 = x_2^{-1}x_1x_2 \qquad y_2 = x_2. \end{aligned} \tag{11}$$

These two elements are seen to generate the inner automorphisms of F_2. For Φ_2 acting on F_2 the kernel ker(h_2) is known to coincide with the group of inner automorphisms of F_2 [1]. In turn, the group of inner automorphisms of F_2 is easily shown to be isomorphic to F_2 acting by conjugations. Therefore we obtain:

Proposition 4. The group ker(h_2) is the normal subgroup $\mathcal{F}_2$ of Φ_2 generated by $\langle t_1, t_2\rangle$ (equation (5)) and is isomorphic to F_2.

The abstract conjugation transformations of the generators of this normal subgroup under the generators of Φ_2 were given in equation (6). With $(t_1, t_2) \leftrightarrow (x_1, x_2)$, the correspondence of the action by conjugations (6) on (t_1, t_2) to the action by automorphisms (10) on (x_1, x_2) is evident. For the multiplication of these transformations note that from equation (6) we compose conjugations whereas from equation (10) we must compose automorphisms according to Nielsen [1].

Now we shall represent F_2 by a graph suggested by the Fricke–Klein geometry [6, 7], and interpret the relations obeyed by Φ_2 in terms of this graph.

Consider a 2D quadratic or linear surface $\mathcal{S}$ in R^3, lines on $\mathcal{S}$ formed from intersections with planes through a fixed point P_0 outside $\mathcal{S}$, and directed segments on these lines. We require that any two distinct points on $\mathcal{S}$ together with P_0 fix a plane and a line. These properties apply to the following particular geometries. For $SU(2)$ geometry, $\mathcal{S}$ is the unit sphere around the origin P_0 and the segments are directed arcs on great circles. For $SU(1, 1)$ geometry, $\mathcal{S}$ is one of the three unit hyperboloids in which the planes pass through the origin P_0 and define hyperbolic segments. For planar geometry, $\mathcal{S}$ is a plane and P_0 is a point not in $\mathcal{S}$. Given one of these geometries, we choose two fixed intersecting lines and associate the generators $\langle x_1, x_2\rangle$ of F_2 with two directed segments or paths on these lines with the convention that the segment may be moved on the line. We interpret multiplication in F_2 by path concatenation and inversion by a change of direction. Define x_3 by $x_3 = (x_1x_2)^{-1}$, which is a well defined line segment, so that $x_1x_2x_3 = e$ is a closed directed path around a triangle T. We denote each vertex of this triangle by the number of the opposite path.

The group Φ_2 acts as Aut(F_2) on F_2 and must transform the triangle T into its image T'. We shall discuss now these transformations in the path geometry and examine in particular the relations $Q_1, \ldots, Q_6$. In figure 1 we represent the action of the generators c_2, c_3, σ_1 on

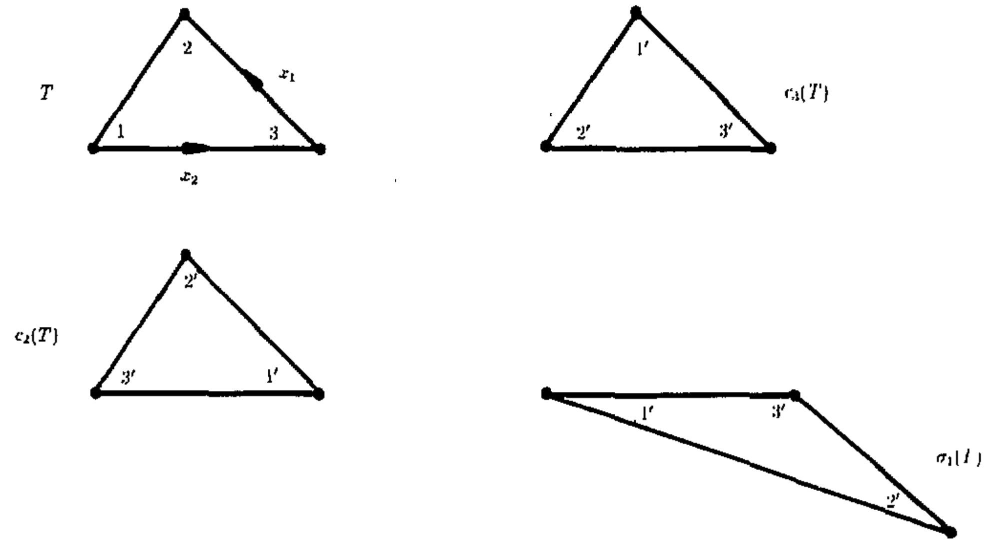

Figure 1. Images $T' = g(T)$ of the triangle T under the involutions c_2, c_3, σ_1.

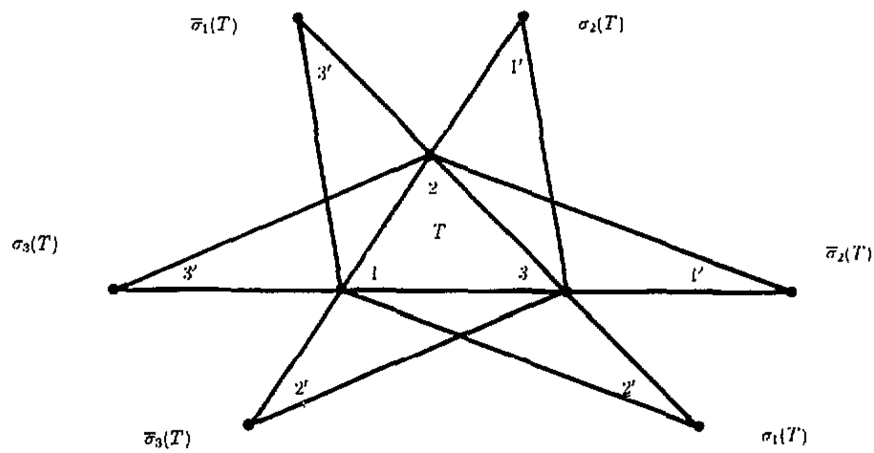

Figure 2. Images $T' = g(T)$ of the triangle T under six conjugations of σ_1 with the group generated by $\langle c_2, c_3\rangle$.

the triangle in the planar geometry. The first two generators simply permute pairs of vertices of T. The Coxeter group $A_2 = \langle c_2, c_3\rangle$ determined by Q_1, Q_2, Q_4 consists of all vertex permutations of T. The generator σ_1 yields a new triangle T' which shares an edge with T and has a new vertex $2'$ on the line passing through vertices 2, 3. This is an involution in agreement with Q_3. By conjugation with the six elements of A_2 we obtain six involutions similar to σ_1 which are shown in figure 2. The powers of $(\sigma_1 c_3), (\sigma_1 c_3)^4 = e$ (relation Q_5), are shown in figure 3—they generate a parallelogram. The Coxeter group generated by $\langle c_3, \sigma_1\rangle$ maps this parallelogram into itself. The element t_2 (equation (5)), generated as a square of an automorphism, is shown in figure 4. Together with t_1 it generates discrete 'parallel transports' of T along lines passing through the edges 1, 2 of the original triangle.

The distance of the parallel transport is twice the length of the line segment. The relation Q_6 implies that a reflection σ preserving this edge of T commutes with parallel transport of T along this line.

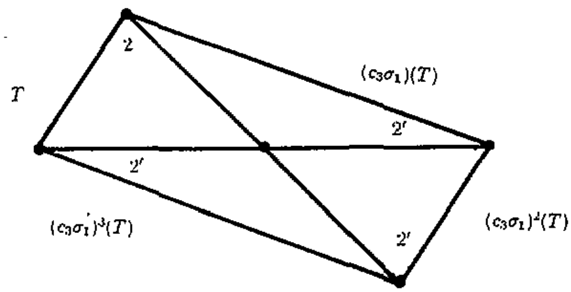

Figure 3. Images $T' = g(T)$ of the triangle T under powers of the automorphism $(c_3\sigma_1)$.

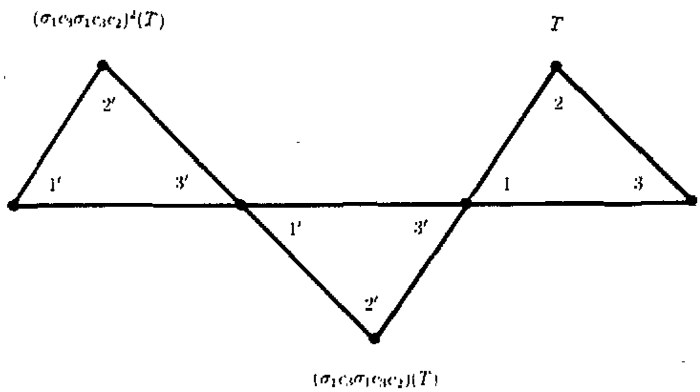

Figure 4. Parallel transport of the triangle T by the square t_2 of the automorphism $(\sigma_1 c_3\sigma_1 c_3 c_2)$.

3. Planar geometry and the homomorphism $h_2 : \Phi_2 \to Gl(2, Z)$

In this section we examine the action of Φ_2 in the planar geometry in terms of certain reflections. We show that the homomorphism $h_2 : \Phi_2 \to Gl(2, Z)$ appears in this geometry and is generated by three non-Weyl reflections.

Let r be a vector in R^3 and ϕ_r a linear form with the property $\phi_r(r) = -2$. The linear map $R^3 \to R^3$

$$(r, \phi_r) : x \to x' = x + \phi_r(x)r \tag{12}$$

is easily shown to be an involution. All points y of the plane $\phi_r(y) = 0$ are stable under this involution.

Proposition 5. Let ξ^1, ξ^2, ξ^3 be linearly independent unit vectors in R^3. The action of Φ_2 on F_2 in the plane containing the three points ξ^1, ξ^2, ξ^3 is generated by three involutions of the type given in equation (12).

Proof. It suffices to construct the three involutions for the generators $\langle c_2, c_3, \sigma_1\rangle$ of Φ_2. We specify three pairs (r, ϕ_r), where each linear form is fixed by giving its value for three vectors:

$$\begin{aligned} c_2 &: r = (\xi^3 - \xi^1) && \phi_r : \phi(\xi^2) = \phi(\tfrac{1}{2}(\xi^3 + \xi^1)) = 0 && \phi(\xi^3 - \xi^1) = -2 \\ c_3 &: r = (\xi^1 - \xi^2) && \phi_r : \phi(\xi^3) = \phi(\tfrac{1}{2}(\xi^1 + \xi^2)) = 0 && \phi(\xi^1 - \xi^2) = -2 \\ \sigma_1 &: r = (\xi^2 - \xi^3) && \phi_r : \phi(\xi^1) = \phi(\xi^3) = 0 && \phi(\xi^2 - \xi^3) = -2. \end{aligned} \tag{13}$$

From them we compute, with the help of equation (12), the action on the three vectors (ξ^1, ξ^2, ξ^3) and obtain for their images

$$(\zeta^1, \zeta^2, \zeta^3) = (\xi^1, \xi^2, \xi^3)D(q)$$

$$D(c_2) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad D(c_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D(\sigma_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}. \tag{14}$$

These linear maps transform $R^3 \to R^3$, transform the affine plane S containing the points ξ^1, ξ^2, ξ^3 into S and also yield the correct images of the three vectors according to the path construction of section 2. □

The similarity to the spherical and hyperbolic cases [6,7] may be seen from the real versions of equation (35) given below: Write equation (35) in terms of the three vectors and use equation (24) to obtain reflections similar to equation (14). In contrast to equation (14), the reflections (35) are not given as actions $R^3 \to R^3$ which conserve the surface S.

We now add some comments on the non-Weyl reflections. Given a fixed global scalar product $\langle\,,\rangle$ on R^3, we could choose $\phi_r(x) = -2\langle x, r\rangle/\langle r, r\rangle$. Then equation (12) would become a Weyl reflection. In the present case the involution $c_1 = c_2c_3c_2$ has the same vector as σ_1 but differs in the linear form ϕ (see figure 5). It is impossible to describe both maps with a single global metric and we are forced to use non-Weyl reflections. For comparison with the Gram construction in a Coxeter group, we compute the matrix M with entries $-\frac{1}{2}\phi_{r_i}(r_j)$ and obtain

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{15}$$

Clearly $\det(M) = 0$, so that the present affine construction resembles the representation of an affine Coxeter group.

The generators (t_1, t_2) of $\ker(h_2)$ are represented by two commuting translations in the affine plane. We display the action within the affine plane by introducing the relative vectors $(x^1, x^2) = (\xi^2 - \xi^3, \xi^3 - \xi^1)$. These vectors are transformed with the 2×2 subrepresentation d into

$$d(c_2) = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad d(c_3) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad d(\sigma_1) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{16}$$

These matrices are precisely the images of the generators (equation (10)) under the homomorphism $h_2 : \Phi_2 \to Gl(2, Z)$. We summarize the result:

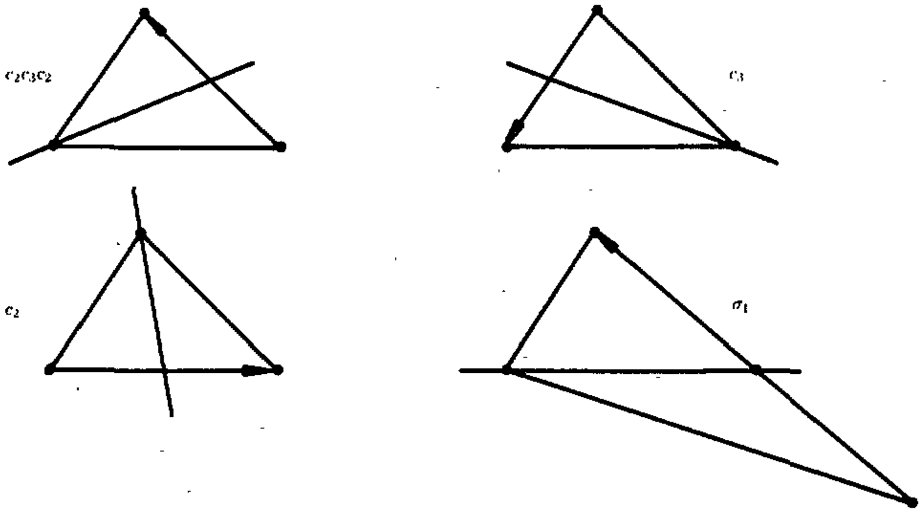

Figure 5. The vectors and reflection lines for the three generating involutions $\langle c_2, c_3, \sigma_1\rangle$ and for $c_1 = c_2c_3c_2$ in the affine plane.

Proposition 6. The group Φ_2 in R^3 has a (linear) representation D generated by three affine non-Weyl reflections in R^3. The normal subgroup $\ker(h_2)$ is represented by commuting affine translations. A subrepresentation d of Φ_2 yields the homomorphism $h_2 : \Phi_2 \to Gl(2, Z)$.

In figure 5 we indicate the pairs (r, ϕ_r) by vectors and reflection lines for the three generators $\langle c_2, c_3, \sigma_1\rangle$. In figure 6 we give an initial triangle T, its images under transformations of the type $(c_3\sigma_1)^n$, and some of their congruent or mirror images. Clearly there are more non-congruent images. The full pattern is symmetric under translations by twice the segment length of x_1, x_2, respectively. The reflection lines of figure 5 generate elements of order three and four and an apparent element of order six which is in $Gl(2, Z)$ but not in Φ_2.

4. Reflections in $Sl(2, C)$

A homomorphism $F_2 \to Sl(2, C)$ is specified by a map $(x_1, x_2) \to (g_1, g_2), g_1, g_2 \in Sl(2, C)$. In sections 4–6 we study the action of Φ_2 on $Sl(2, C)$ induced by this homomorphism. As in [11] we describe pairs of elements of the group $Sl(2, C)$ in terms of three unit vectors. Here we introduce new algebraic reflections generated by these vectors and express the elements of $SL(2, C)$ as products of these reflections. The commutator of two elements of $Sl(2, C)$ is given in terms of these vectors.

We shall use the standard complex scalar and vector products for $SO(3, C)$. Consider elements g of $Sl(2, C)$ and their exponential parametrization

$$g = \exp(-\theta\tilde{\eta}) = \zeta\sigma_0 - \rho\tilde{\eta}$$
$$\tilde{\eta} = \sum_l \eta_l\sigma_l \tag{17}$$

where $\zeta = \cosh\theta$, $\rho = \sinh\theta$, η is a unit vector, $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices and σ_0 is the unit matrix in two dimensions. In sections 4 and 5 we shall use the standard symbols

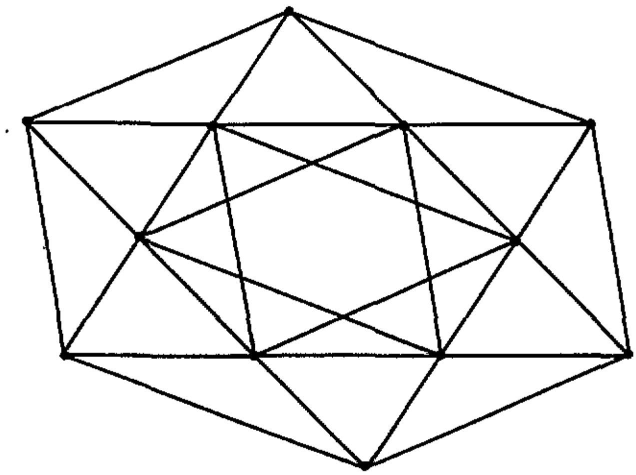

Figure 6. Affine geometry: the triangle T, its images under elements of type $(c_3\sigma_1)^n$ and some of their congruent or mirror images are shown. The full pattern is symmetric under translations by twice the line segments x_1, x_2, respectively. The affine reflections generate elements of order three with centres at midpoints of triangles and elements of order four with centres at midpoints of parallelograms. The apparent six-fold symmetry at the centre occurs as an element of order six in $Gl(2, \mathrm{Z})$ but not in Φ_2.

σ_l for the Pauli matrices since the distinction from the generators of Φ_2 should be clear in any expression. Given a pair of matrices g_1, g_2, we define $g_3 = (g_1 g_2)^{-1}$. It is shown in [11] that any pair g_1, g_2 determines three unit vectors

$$\xi^k \propto (\epsilon_{ljk})^2 (\eta^l \times \eta^j). \tag{18}$$

Construct from the vectors ξ^l the matrices $\tilde{\xi}^l$, called reflections, and observe

$$(\tilde{\xi})^2 = \sigma_0 \tag{19}$$

$$\det(\tilde{\xi}) = -1 \tag{20}$$

$$\tilde{\xi}^1 \tilde{\xi}^2 = (\xi^1 \cdot \xi^2)\sigma_0 + \mathrm{i} \sum_l (\xi^1 \times \xi^2)_l \sigma_l. \tag{21}$$

So the matrices $\tilde{\xi}$ belong to the subgroup of $Gl(2, C)$ with determinant ± 1 and not to $Sl(2, C)$.

Proposition 7. The elements, $g_1, g_2, g_3, g_1 g_2 g_3 = e$, have the decomposition

$$g_1 = \tilde{\xi}^2 \tilde{\xi}^3 \qquad g_2 = \tilde{\xi}^3 \tilde{\xi}^1 \qquad g_3 = \tilde{\xi}^1 \tilde{\xi}^2. \tag{22}$$

Proof. First we construct from g_1, g_2 the unit vector $\xi^3 \propto (\eta^1 \times \eta^2)$ by normalizing the vector product to unit length, and from it we obtain $\tilde{\xi}^3$. Clearly $(\xi^3 \cdot \eta^1) = (\xi^3 \cdot \eta^2) = 0$. For reflections $\tilde{\xi}, \tilde{\eta}$ with $(\xi \cdot \eta) = 0$ and $g = \exp(-\theta \tilde{\eta})$ we find that $g\tilde{\xi}$ and $\tilde{\xi} g$ are reflections. Now from g_2, g_3 we construct

$$\tilde{\xi}^1 := \tilde{\xi}^3 g_2 \qquad \tilde{\xi}^2 := \tilde{\xi}^3 g_2 g_3 = \tilde{\xi}^3 g_1^{-1} \tag{23}$$

to obtain the result equation (22). □

The matrix form of the reflection simplifies the determination of the vectors ξ^i from the group elements g_1, g_2, g_3 [11]. From the expressions (22) we can easily verify that $g_1g_2g_3 = \sigma_0$.

Proposition 8. Let ξ, η be unit vectors. Define the map

$$(\tilde{\xi}, \tilde{\eta}) \to \tilde{\eta}' = \tilde{\xi}\tilde{\eta}\tilde{\xi} = -\tilde{\eta} + 2(\xi \cdot \eta)\tilde{\xi}. \tag{24}$$

The corresponding adjoint action $(\tilde{\xi}, g) \to g'$

$$\mathrm{Ad}_{\tilde{\xi}}(g) = g' = \tilde{\xi}(\zeta\sigma_0 - \rho\tilde{\eta})\tilde{\xi} = \zeta\sigma_0 - \rho\tilde{\eta}' \tag{25}$$

is an involution in $Sl(2, C)$. The factorization (22) of group elements from $Sl(2, C)$ into products of two matrices of the type $\tilde{\xi}$ generates these group elements as products of two reflections. The adjoint action is obtained in the form $\mathrm{Ad}_{g_3} = \mathrm{Ad}_{\tilde{\xi}_1} \mathrm{Ad}_{\tilde{\xi}_2}$.

Proposition 9. The commutator $K(g_2, g_1) := g_2g_1g_2^{-1}g_1^{-1} = g_2g_1g_3$, when expressed in terms of the vectors ξ, has the form

$$K = (\tilde{\xi}^3\tilde{\xi}^1\tilde{\xi}^2)^2$$
$$(\tilde{\xi}^3\tilde{\xi}^1\tilde{\xi}^2) = \mathrm{i}(\xi^1 \times \xi^2) \cdot \xi^3)\sigma_0 + \sum_l \left[(\xi^1 \cdot \xi^2)\xi^3 - (\xi^2 \cdot \xi^3)\xi^1 + (\xi^3 \cdot \xi^1)\xi^2\right]_l \sigma_l. \tag{26}$$

The commutator with $\Delta := (\xi^1 \times \xi^2) \cdot \xi^3$ becomes

$$K = (1 - 2\Delta^2)\sigma_0 + 2\mathrm{i}\Delta \sum_l \left[(\xi^1 \cdot \xi^2)\xi^3 - (\xi^2 \cdot \xi^3)\xi^1 + (\xi^3 \cdot \xi^1)\xi^2\right]_l \sigma_l. \tag{27}$$

Specific results for the subgroups $SU(2)$ and $SU(1, 1)$ are obtained by appropriate restrictions of the parameters: for $SU(2)$ we put $\theta = i\alpha$ and choose α and the vectors ξ, η to be real. The complex unit sphere reduces to the real unit sphere S_2 in the geometry of $SO(3, R)$. The results for $SU(2)$ are closely related to the theory of turns treated by Biedenharn and Louck [12] (cf [7]).

For $SU(1, 1)$, we use the vector $\eta = (q_1, -q_2, \mathrm{i}q_3)$ with real components q_i and replace the Pauli matrices according to $\sigma_1' = \sigma_1$, $\sigma_2' = -\sigma_2$, $\sigma_3' = \mathrm{i}\sigma_3$ [6]. The complex unit sphere reduces to one of the unit hyperboloids in the geometry of $SO(2, 1, R)$. There are relations to the geometry of Fricke and Klein [13] (cf [6]) and to work by Vogt [14]. The three vectors ξ^l are always on a single hyperboloid (cf [6, 7]).

5. Matrix products in $Sl(2, C)$ and reflections

In applications [6] one often generates words in F_2 by the action of elements from Φ_2. The induced action of Φ_2 on $Sl(2, C)$ generates matrix products in $Sl(2, C)$. A standard form of these matrix products would be helpful for these applications. Standard forms for the traces of these words are treated in the ring theory of Fricke characters [13–16]. Applications in physics, for example, in the 1D S-matrix problem [6, 8], require the knowledge of the full matrix image under the induced action. In the present section we use the reflections introduced in section 4 to express matrix products from n elements of $Sl(2, C)$ as linear combinations of fundamental matrices.

Let $\xi^1, \ldots, \xi^{n+1}$ be a general set of complex unit vectors in the $SO(3, C)$ metric.

Definition 1. The 2^{n+1} fundamental ascending $\tilde{\xi}$-products are

$$\sigma_0, \prod \tilde{\xi}^{\mu_1} \ldots \tilde{\xi}^{\mu_r}, \mu_1 < \mu_2 \ldots < \mu_r \qquad 1 \leqslant r \leqslant n+1. \tag{28}$$

Proposition 10. Any product $\prod'$ of degree q formed from the matrices $\tilde{\xi}^j$ can be written as a linear combination of the fundamental matrix products (28). The linear coefficients are polynomials in the scalar products $(\xi^i \cdot \xi^j), i < j$ with integral coefficients.

Proof. For any descending pair of subsequent matrices in $\prod'$ we apply equation (21) in the form

$$s > t : \tilde{\xi}^s \tilde{\xi}^t = -\tilde{\xi}^t \tilde{\xi}^s + 2(\xi^t \cdot \xi^s)\sigma_0. \tag{29}$$

Substitution in $\prod'$ yields the ascending order for this pair and introduces an additional matrix product term of degree $q-2$, where the pair is replaced by twice the scalar product. A finite number of these steps leads to an ascending order in all matrix terms. □

We pass from the $n+1$ reflections to n elements h_i of $Sl(2, C)$. We use the letters h_i rather than g_i since their indexing differs from the one used in equation (22).

Proposition 11. Let $h_i, i = 1, 2, \ldots, n$, be general elements of $Sl(2, C)$. There exist $n+1$ reflections $\tilde{\xi}^j, j = 1, 2, \ldots, n+1$, so that

$$h_i = \tilde{\xi}^i \tilde{\xi}^{i+1} \qquad i = 1, \ldots, n. \tag{30}$$

Proof. We assume that the unit vectors η^{i-1}, η^i, which generate $h_{i-1}, h_i, i = 2, \ldots, n$, are linearly independent and define by normalization up to a sign

$$\xi^i \propto \eta^{i-1} \times \eta^i \qquad i = 2, \ldots, n. \tag{31}$$

Fixing a sign for ξ^2, we determine ξ^1 from the reflection $\tilde{\xi}^1 := h_1 \tilde{\xi}^2$ to obtain $h_1 = \tilde{\xi}^1 \tilde{\xi}^2, \eta^1 \propto \xi^1 \times \xi^2$. Now, from equation (31), $\xi^2 \times \xi^3 \propto (\eta^1 \times \eta^2) \times (\eta^2 \times \eta^3) \propto \eta^2$ and so we may choose the sign of ξ^3 from $\tilde{\xi}^3 = \tilde{\xi}^2 h_2$ to obtain $h_2 = \tilde{\xi}^2 \tilde{\xi}^3$. Continuing in this fashion we fix all the signs and get the result, equation (30). □

Consider now a general product $\prod'$ formed from $h_1 \ldots h_n \in Sl(2, C)$.

Definition 2. The 2^n ascending fundamental h-products are

$$\sigma_0, \prod h_{\nu_1} \ldots h_{\nu_k}, \nu_1 < \nu_2 \ldots < \nu_k \qquad 1 \leqslant k \leqslant n. \tag{32}$$

Proposition 12. Any product $\prod'$ of degree p formed from n matrices $h_j \in Sl(2, C)$ can be written as a linear combination in the 2^n ascending fundamental matrix products of the h_j (32). The linear coefficients are polynomials in the expressions $\frac{1}{2}\operatorname{tr}(h_i h_{i+1} \ldots h_{i+q-1}), q > 1$, with integral coefficients.

Proof. We rewrite $\prod'$ by use of equation (30) as an even product of the $n+1$ reflections $\tilde{\xi}^i$. By applying proposition 11, it can be expressed as a linear combination in the even fundamental ascending products of the reflections. Now we observe that from equation (30) for $q > 1$

$$\tilde{\xi}^i \tilde{\xi}^{i+q} = h_i h_{i+1} \dots h_{i+q-1}. \tag{33}$$

It follows from this equation that any even ascending matrix term in the $\tilde{\xi}^i$ can be replaced by an even or odd ascending matrix term in the h_j. Moreover the coefficients in the linear combinations may be rewritten in terms of the h_j by noting from equation (33) that

$$(\xi^i \cdot \xi^{i+q}) = \tfrac{1}{2}\,\mathrm{tr}(h_i h_{i+1} \dots h_{i+q-1}) \qquad q > 1. \tag{34}$$

□

This proposition generalizes the results of Fricke [3, 15, 16], from the level of characters or traces to the level of matrices.

6. Φ_2 acting on $Sl(2, C)$

Let $(x_1, x_2) \to (g_1, g_2)$ be a homomorphism from the free group F_2 to $Sl(2, C)$, and let $\Phi_2 = \mathrm{Aut}(F_2)$ act on the images (g_1, g_2). We shall describe this action with the help of the vectors introduced in section 4. The new generators of Φ_2 and the algebraic treatment of reflections yield a new and simplified form of the results given in [6, 7, 11].

We showed in section 2 that Φ_2 is generated by the three involutions c_2, c_3, σ_1.

Proposition 13. The generators of Φ_2 yield, with respect to the matrices $\tilde{\xi}$, the transformations

$$\begin{aligned} c_2 &: (\tilde{\xi}^1, \tilde{\xi}^2, \tilde{\xi}^3) \to (\tilde{\xi}^3, \tilde{\xi}^2, \tilde{\xi}^1) \\ c_3 &: (\tilde{\xi}^1, \tilde{\xi}^2, \tilde{\xi}^3) \to (\tilde{\xi}^2, \tilde{\xi}^1, \tilde{\xi}^3) \\ \sigma_1 &: (\tilde{\xi}^1, \tilde{\xi}^2, \tilde{\xi}^3) \to (\tilde{\xi}^1, \tilde{\xi}^3\tilde{\xi}^2\tilde{\xi}^3, \tilde{\xi}^3). \end{aligned} \tag{35}$$

The first two generators yield transpositions and through them generate the Coxeter group A_2. The last generator is expressed by a reflection of one of the three vectors (24). The action (35) of the group Φ_2 on the three reflections has an exact correspondence to the abstract action of Φ_2 by conjugation on the three involutive generators of $\mathcal{H}_2$ obtained in equation (8).

By Nielsen's theorem [2], under any automorphism of F_2 the commutator is transformed into a conjugate of itself or of its inverse. The explicit form (26), (27) of K allows us to study this transformation in detail. For the traces it is easy to see from equation (26) that, under any one of the generators equation (35) of Φ_2, the quantity $\Delta = -(\mathrm{i}/2)\,\mathrm{tr}(\tilde{\xi}^3\tilde{\xi}^1\tilde{\xi}^2)$ is multiplied by a factor (-1). Hence the volume spanned by the three vectors is conserved up to a sign under Φ_2, and the usual trace invariant [6] is, from equation (27), $\frac{1}{2}\,\mathrm{tr}(K) = 1 - 2\Delta^2$. Various applications in physics of actions induced from Φ_2 to trace and in particular to matrix systems can be treated efficiently with the methods given in sections 4–6.

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References

[1] Nielsen J 1924 *Math. Ann.* **91** 169–209
[2] Magnus W, Karrass A and Solitar D 1976 *Combinatorial Group Theory* 2nd edn (New York: Dover)
[3] Brown H, Bülow R, Neubüser J, Wondratschek H and Zassenhaus H 1978 *Crystallographic Groups of Four-Dimensional Space* (New York: Wiley)
[4] Kramer P 1993 *Anales de Fisica* vol 1, ed M A Del Olmo and M Santander (Madrid: CIEMAT/RSEF) p 370
[5] Garcia-Escudero J and Kramer P 1993 *Anales de Fisica* vol 1, ed M A Del Olmo and M Santander (Madrid: CIEMAT/RSEF) p 339
[6] Kramer P 1993 *J. Phys. A: Math. Gen.* **26** 213–28
[7] Kramer P 1993 *Free Groups, Their Automorphism Groups, Applications (Symmetries in Science VI)* ed B Gruber *et al* (New York: Plenum)
[8] Baake M, Joseph D and Kramer P 1992 *Phys. Lett.* **186A** 199–208
[9] Baake M, Grimm U and Joseph D 1993 *Int. J. Mod. Phys.* B **7** 1527–50
[10] Peyriere J 1991 *J. Stat. Phys.* **62** 411–4
[11] Kramer P 1993 *J. Phys. Lett.* A **26** L245–50
[12] Biedenharn L C and Louck J D 1981 *Angular Momentum in Quantum Physics: Theory and Applications* (Reading, MA: Addison-Wesley)
[13] Fricke R and Klein F 1897 *Vorlesungen über die Theorie der Automorphen Funktionen* vol 1 (Leipzig: Teubner)
[14] Vogt M H 1889 *Ann. Sci. Ecole Norm. Sup.* **6** 3–71
[15] Magnus W 1980 *Math. Z.* **170** 91–103
[16] Horowitz R 1975 *Trans. Am. Math. Soc.* **208** 41–50